constant average value of a required to use the same amount of fuel in the same transfer time as would a constant a program For transfer between given coplanar circular orbits in a specified time, $\bar{a}$ is always less than or equal to the required constant a to accomplish the same transfer The altitude gain parameter is plotted in Fig 7 for both modes, constant- and variable-thrust acceleration As anticipated, the variable acceleration curve yields superior performance for all transfer times except integer multiples of the orbital period of the reference orbit At these times, constant-thrust acceleration is optimal, and the curves are tangent to one another

Conclusions

1) The coplanar solutions for values of transfer time equal to an integral multiple of the orbital period are found to reduce to constant circumferentially directed acceleration

2) At other values of transfer time, the optimum thrust magnitude program is shown to yield better performance than the constant-acceleration program

3) The optimal change in angular circumferential position is a linear function of transfer time

4) For all but short transfer times, the restriction on the velocity components imposed by linearization can be lifted

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Surface Strains in Case-Bonded Models of Rocket Motors

D J Bynum,* L U Rastrelli,† and R C DeHart‡

Southwest Research Institute, San Antonio, Texas

Strain measurements on the case of rocket engine models with applied loads were made with electric-resistance-type strain gages, enabling the computation of case surface stresses The model consisted of an aluminum case with a bonded, viscoelastic grain A model having a circular-shaped grain was tested first with an applied internal pressure and then was tested to failure with an applied axial load, uniformly distributed A model having a star-shaped grain was loaded with an applied internal pressure until a failure by rupturing occurred

Introduction

THE reported tests were undertaken in order to quantitatively ascertain the allowable loads on model rocket engines tested in an extensive, exploratory program in which the internal displacements in solid propellant grains were experimentally determined by observing embedded particles with an x-ray scintillation detection facility [1–3] Models cast from inert and live grains but without cases, had been successfully tested previously using the x-ray facility The accuracy of the x-ray particle detection system was thoroughly investigated in another study [4]

The model rocket engines consisted of an aluminum case and a viscoelastic grain The grain material was an inert polyurethane that had mechanical properties similar in nature to some of the commonly used solid propellants

One of the models had a circular-shaped grain and was tested first with an applied internal pressure and then was tested to failure with an applied axial load, uniformly distributed The other model had a star-shaped grain and was tested with an applied internal pressure until a failure of the grain by rupture occurred

The particular type of end restraints imposed on the models consisted of a platen arrangement whereby the ends of the models were free to translate longitudinally but were restrained in the radial direction Electrical-resistance-type strain gages bonded to the outer surface of the cases were used for measuring the applied strains, from which the case stresses were calculated The gages were metal film, type C6-111§ having a $\frac{1}{16}$-in gage length, a 2 01 ± 1% gage factor, and a 120 ± 0 5-ohm resistance Dummy gages, mounted on a sample of the same type of material from which the model cases were fabricated, were used for temperature compensation, since the C6 gage is self-temperature compensated for use only on steel The strain gage installations are shown in Fig 1

Experimental Procedure

Internal Pneumatic Pressure Tests of Model Having a Circular-Shaped Grain

The model was subjected to internal pneumatic pressures of 0 to 160 psig in 20-psig increments The Baldwin-Tate

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* Research Engineer, Department of Structural Research

† Research Manager, Department of Structural Research

‡ Director, Department of Structural Research

§ Available from the Budd Company, P O Box 245 Phoenixville, Pa

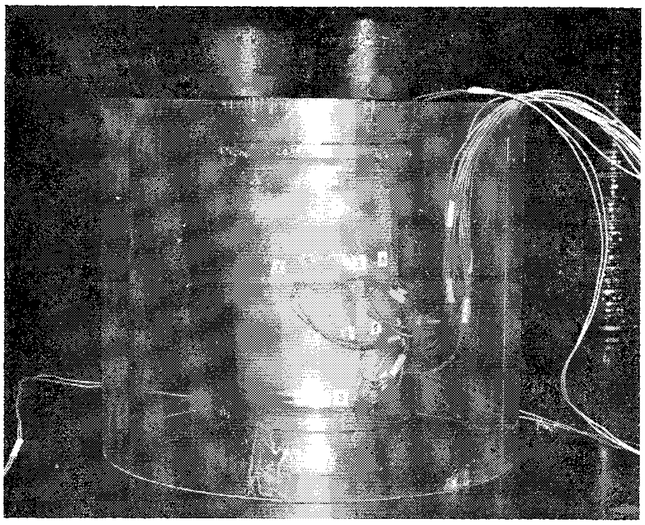

Fig 1 Model having star-shaped grain with internally applied pressure

Emery (BTE) 200-kip Universal testing machine was used to restrain longitudinal displacement of the platens when internal pressure was applied in the model Three lugs of equal length were positioned on the top platen in the manner shown in Fig 1 to provide access to the pressure fitting that was located in the center of the top platen head The head of the BTE was then lowered on the lugs and adjusted so that the axially applied load was initially zero

Two strain gages were mounted on the case at the center of the model to measure hoop and meridional strains An oscillograph,¶ in conjunction with a bridge balance unit,** was used to record the indicated strains The feed rate of the recording paper was 0 2 in /min, and the trace calibration was 1 in /1000 μin /in The time period for the tests was 50 min

Axially Applied Compressive Tests of Model Having a Circular-Shaped Grain

After the internal pressure tests, the model having a circular-shaped grain was subjected to uniformly distributed, axially applied, compressive loads with the BTE testing machine The loads were applied in 500-lb increments from zero load to the failure load in a time period of 33 min A strain indicator†† was used to measure the indicated strains

Internal Hydraulic Pressure Tests of Model Having a Star-Shaped Grain

The model having a star-shaped grain was subjected to internal hydraulic, instead of pneumatic, pressure so that high values of applied pressure could be safely used Eight strain gages were mounted on the case of the model, and the indicated strains were recorded with a strain gage plotter ‡‡ The model was subjected to pressure in 50-psig increments from zero to failure pressure

Details of Models

Platen Assembly

A platen assembly was used on each end of the models so that internal pressure or a uniformly distributed axial load could be applied The platen assembly consisted of a steel ring, a steel head, and a Buna-N rubber "O" ring, and the platen ring was machined to a dimension so that the ring was a running fit on the model The assembled platens are shown in Fig 1, and the individual parts of the platens are shown in Fig 2 When the model and platens were assembled, the ends of the model were flush against the steel head, and the "O" ring was compressed between the chamfer on the steel ring, the case of the model, and the platen head This arrangement provided an efficient seal for values of internal pressures up to rupture pressure

Model Geometry

The geometry of the models tested are given in Figs 3 and 4 for the star-shaped and the circular-shaped grains, respectively The star model was 6 13 in in length, and the model with the circular-shaped grain was 7 00 in in length, and so the ratios of the lengths divided by the outside case diameter were 2 01 and 2 33 The load platens enclosed about 0 75 in of the specimen on each end, and so the effective model length, the distance between the faces of the platen ring, was approximately 1 50 in less than the actual model length The effective length to diameter ratios were then

$$(L'/D)_{ic} = (7\ 00 - 1\ 50)/3\ 00 = 1\ 83 \text{ in}$$

$$(L'/D)_{ta} = (6\ 13 - 1\ 50)/3\ 00 = 1\ 54 \text{ in}$$

These ratios were reasonably well-modeled proportions, since ratios of L/D for rocket engines vary, in general, between 1 5 and 3 5

An "O"ring having a $\frac{1}{8}$-in wall thickness was used with the platens for the tests of the model with the circular grain This ring was effective only for low values of internal pressure, and so $\frac{1}{8}$-in length of the case was machined from each end of the model, thereby enabling larger values of applied pressure to be used The ends of the case and the grain of the star model were machined flush, and an effective seal for large values of applied pressure was obtained by using an "O" ring having a $\frac{3}{16}$-in wall thickness

Grain Casting Procedure

An inert formulation was used for the grain, and the composition was similar to a currently used solid propellant The grain binder was Adiprene L §§ Each of the grain constituents was preheated to 180°F, mixed at this elevated temperature, and then degassed in a vacuum oven at 180°F The mix was then poured into an aluminum cylinder, which was positioned in the casting jig and cured in an oven for 2 hr at 285°F The inner surfaces of the aluminum tubes were sandblasted to increase the efficiency of the bond between the tube and the grain

The casting jig consisted of a steel cylinder, machined to a running fit over the case to maintain case roundness, and an aluminum core, centered and fastened to a platen, to create

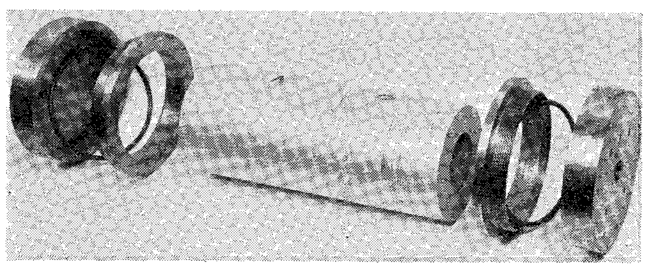

Fig 2 Load platen arrangement

¶ Visicorder Oscillograph, model 906A; available from Honeywell Industrial Products Corporation, 5200 E Evans Ave Denver 22, Colo

** Heiland Bridge Balance Unit, model 82-6; available from Heiland Research Corporation, 130 E Fifth Ave, Denver 9, Colo

†† Baldwin SR-4 Strain Indicator type N; available from the Baldwin-Lima-Hamilton Company, Waltham 54, Mass

‡‡ Gilmore Strain Gage Plotter, model 114; available from the Gilmore Industries Cleveland, Ohio

§§ Trade name for 2 4-tolylene di-isocyanate; available from the E I Dupont de Nemours and Co, Inc, Wilmington 98, Del

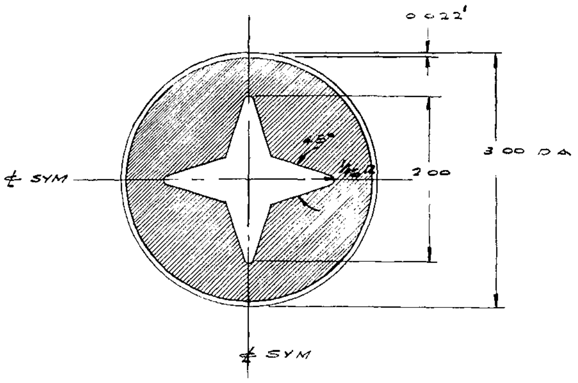

Fig 3 Cross-sectional geometry of specimen having a star-shaped grain

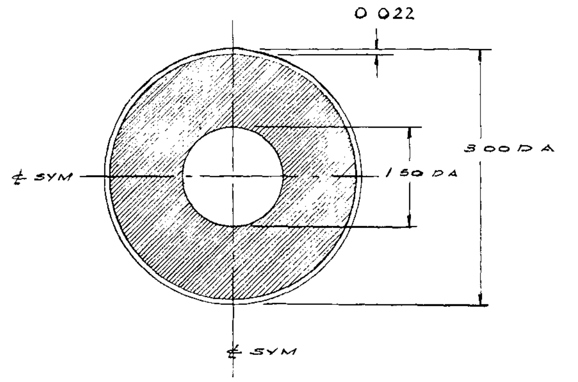

Fig 4 Cross-sectional geometry of specimen having a circular-shaped grain

the required shape of grain chamber After the casting was allowed to cool to room temperature, the steel case was removed, and the aluminum core was forced out

Material Properties

The case material was 6061-T6 aluminum The aluminum was in the form of an extruded tube, the mechanical properties[5] of which were σ_y = 35 ksi, σ = 38 ksi, and elongation (in 2 in or in 4 diam) = 10% A Young's modulus of elasticity of 10 5 × 10^6 psi and a Poisson's ratio of 0 33 were used for the stress calculations from the strain gage readings during the elastic responses of the material

An instantaneous durometer of 63 and a durometer of 48 after stress relaxation (about 1 min) were determined for the cast grains, using a Shore B-2 durometer tester The physical, chemical, and mechanical properties of Adiprene L, given as functions of several parameters including durometer, are presented in the literature[6 7] Since Adiprene L is a viscoelastic material, these functions are nonlinear Some of these functions are also presented in graphical form in a Southwest Research Institute interim report[8]

Discussion of Results

Internal Pneumatic Pressure Tests of Model Having a Circular-Shaped Grain

The Visicorder strain traces indicated a linear relationship for the strains in the case as a function of internal pressure The stress relaxation of the grain could possibly influence the case strains with loads, depending on model geometry, but, in this case, no stress relaxation could be detected

The experimentally determined hoop and meridional strains per unit pressure were ϵ_h = 5 18 and ϵ_m = 1 50 μin / in /psig, respectively

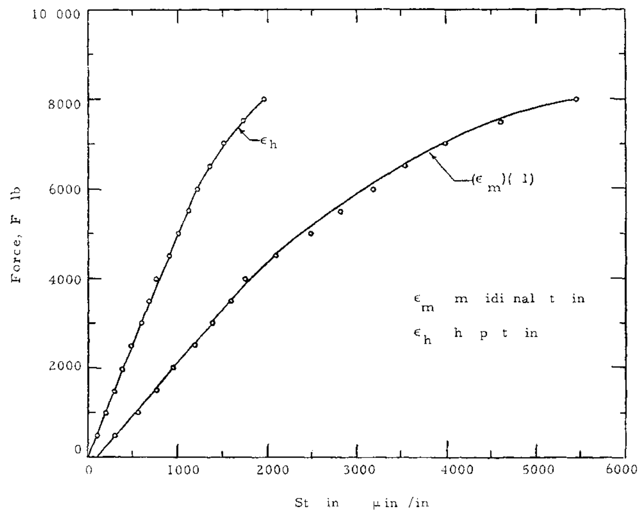

Fig 5 Case strains due to uniformly distributed, compressive loads axially applied on a rocket engine model having a circular-shaped grain

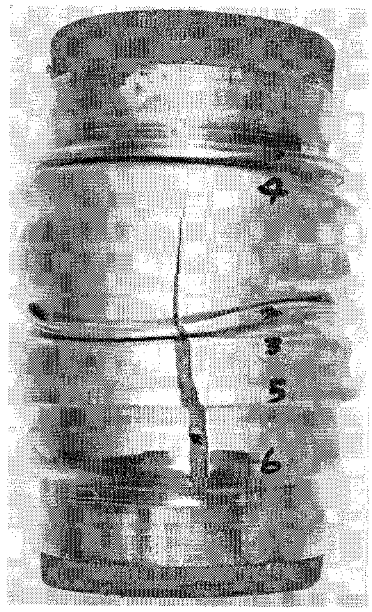

Fig 6 Rupture of model having a circular-shaped grain

The hoop stress was the maximum stress in the case, as would be expected The experimental value of the hoop stress had a value that was approximately midway between the extreme theoretical values computed considering a plane state of stress and a plane state of strain The meridional stress results indicated that the test specimen was close to being in a state of plane stress

Axially Applied Compressive Tests of Model Having a Circular-Shaped Grain

The indicated strains due to the applied loads are shown in Fig 5 From the indicated strains in the elastic range, a meridional stress per unit load of −3 98 psi/lb and a hoop stress per unit load of 0 944 psi/lb were calculated

The model initially failed in a buckling mode After the successive formation of six circumferential rings, the failure order of which is given in Fig 6, the model ruptured along a meridian at a load of 8200 lb After the platens were removed from the specimen, the grain demonstrated its delayed elastic properties by extruding about $\frac{1}{2}$ in out of the case The grain appears to have behaved with characteristics similar in nature to the Kelvin model[9] That is, the grain experienced large deformations without any indication of rupture

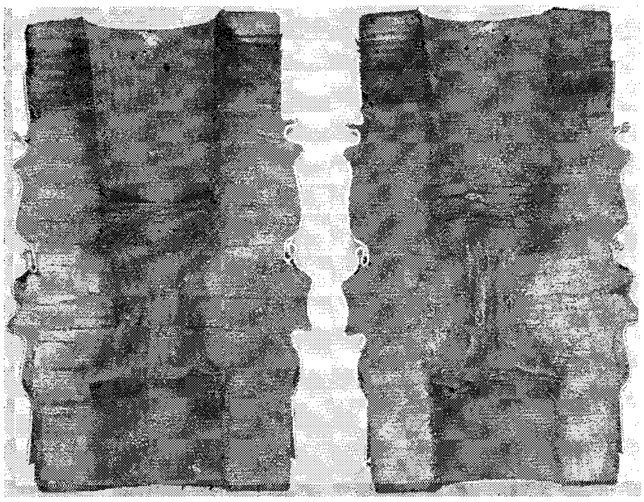

Fig 7 Cross sections of ıuptured model having a cıiculaı-shaped grain

The model was cut in half in order that the plastic flow pattern of the grain could be observed, and the sections are shown in Fig 7 The case rupture is near the center of the specimen section shown on the left side of Fig 7 The grain seemed to be impervious to oil The oil lines appearing in Fig 7, the dark areas along the inner surface of the grain, occurred when the specimen was being cut

Internal Hydraulic Pressure Tests of Model Having a Star-Shaped Grain

The tabulated results for the internal hydraulic pressure tests of the model having a star-shaped grain are given in Table 1 and graphically presented in Fig 8 The maximum hoop stress per unit load, 78 4 psi/psig, occurred at the center of the specimen where the grain thickness was a minimum, as would be expected The nonlinear variation of the hoop stress along the specimen length was because of the radial restraint offered by the platens The meridional stress along the specimen length also varied, but in a linear fashion

The meridional rupture, which occurred at the minimum section, of the model can be seen in Fig 9 The rupture pressure was about 675 psig The specimen was cut in half along the rupture and is pictured in Fig 10 As shown in this figure, the grain also ruptured at the grain fillet tips

Conclusions

The ıesults of the experimentally determined stresses ıeflect the binding action that occurred between the ends of the specimen and the platen rings, and this restraint prevented completely free longitudinal translation of the case with

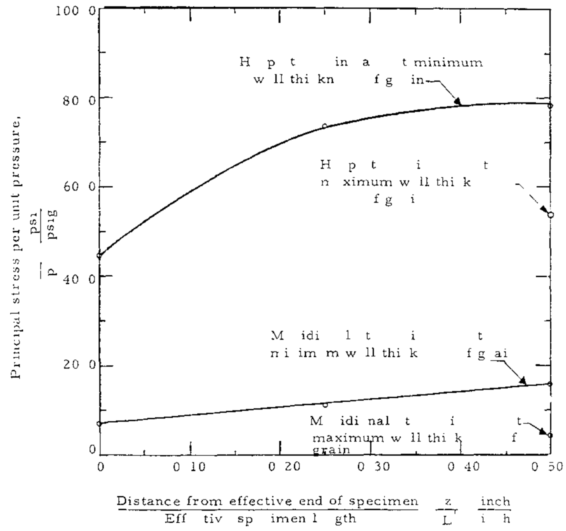

Fig 8 Elastic case stıains due to an applied inteınal pressuıe in a rocket engine model having a star-shaped grain

applied load If the platens could have been designed so that a clearance existed between the platen ring and the specimen with applied load, the experimental meridional stress would be zero, corresponding to a plane state of stress The clearance was restricted because of the seal arrangement If the clearance had been sufficiently decreased causing a frictional force to act on the case and to provide complete restıaint, the experimental meridional stress would correspond to the theoretical meridional stress considering a plane state of strain

As previously mentioned, the reported tests were undertaken for the sole purpose of establishing the modes of failure for uniformly distributed, compressive loads or inteınal pressure applied to case-bonded, solid propellant models of rocket motors The results of these tests were used for the guidance of subsequent tests of similar models wherein the displacements of embedded particles were determined using the x-ray scintillation detection facility

Further tests are tentatively planned for the determination of the whole field variations of the stresses in rocket engine models because of applied axial loads or internal pressure The whole field stresses could be easily obtained with the birefringent coating technique

Other tests planned include the determination of thermal stresses in rocket engine models resulting from thermal transients emanating from the cavity of the model Thermal stresses in the grain will be determined with the embedded particle technique The thermal stresses in the case will be

Table 1 Inteınal pressure test ıesults, star-shaped grain (for elastic responses of case)

Strain gage number[a]	Distance from gage to platen per effective specimen length z'/L', in /in	Hoop strain per unit pressure ϵ_h/p, μ in /in -psig	Meridional strain per unit pressure ϵ_m/p, μ-in /in -psig	Hoop stress per unit pressure σ_h/p psi/psig	Meridional stress per unit pressure σ_m/p, psi/psig
1	0 50		−1 56		4 36
2	0 50	5 84		62 8	
3	0		−0 725		7 08
4	0	4 02		44 6	
5	0 25		−0 409		11 2
6	0 25	6 64		73 5	
7	0 50		−0 305		16 0
8	0 50	6 95		78 4	

Gages numbered 1 and 2 were installed on the case at a circumferential position corresponding to the maximum grain thickness Gages numbered 3–8 were installed on the case at circumferential positions corresponding to the intersection of a case meridian with a radial line that bisected a root of the grain

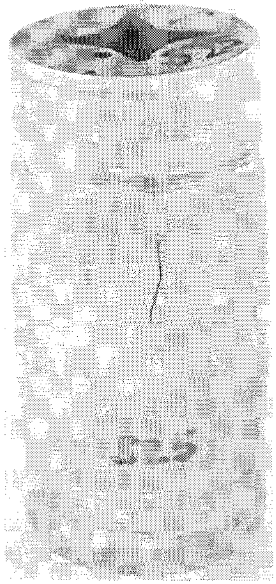

Fig 9 Ruptuie of model having a stai-shaped giain

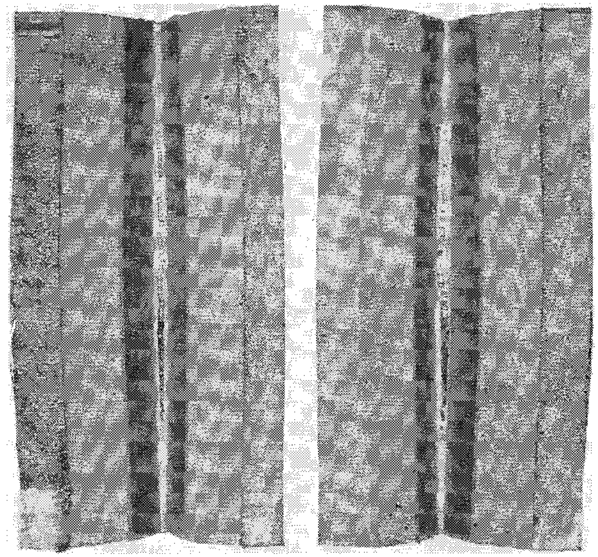

Fig 10 Cioss sections of iupture d model having a stai-shaped giain

determined using electrical-resistance type of strain gages or the birefringent coating technique

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